G - Polish group

G-flow: cHaus X equipped with cont. action $G \times X \to X$

A G-flow X is *minimal* if the only subflow (non-$\emptyset$, closed, G-inv. subset) of X is X itself.

Zorn: every G-flow contains a min subflow.

Two tasks: 1) Understand minimal flows

2) Understand how min subflows sit inside a given flow

Fact (Ellis 1960): $\exists$ a univ. min. flow, a minimal G-flow which admits a G-map (cont., G-equiv) to every min G-flow

It is unique up to iso, denote by $M(G)$

- For any l.c. Polish grp, $M(G)$ is non-metrizable (K.P.T. 05), and the action is free (Veech)

- Other extreme: <u>extremely amenable</u> groups are groups with $M(G) = \{\cdot\}$

Equiv, every G-flow contains a fixed pt.

Ex: Unitary grp (Gromov-Milman 84), $\mathrm{Aut}(X, \mu)$ (Giordano-Pestov 02), $\mathrm{Aut}(\mathbb{Q}, <)$ (Pestov 98)

- $\exists$ Polish grps G with $M(G)$ non-triv., but metrizable.

Ex: G = compact Polish grp, $M(G) \cong G$

$G = \mathrm{Homeo}^+(\mathbb{T})$, ~~~~~~ see chalk

- $\exists$ Polish groups with $M(G)$ containing a comeager orbit, but with $M(G)$ non-metr.

<u>Thm</u>: (Z. for G non-Arch, BYMT 2017)

$M(G)$ metr. $\longrightarrow$ $M(G)$ has comeager orbit.

Ex: various homeo grps related to dendrites.

Q: how can min subflows look like? X a G-flow.

$V(X)$ - the Vietoris space of compact subsets of X

Basic open nbd: $\{Y \in V(X) : Y \subseteq A_0 \cup \cdots \cup A_{k-1}$ and $\forall i < k \; Y \cap A_i \neq \emptyset\}$

$A_i \subseteq X$ are open.

$\mathrm{Sub}_G(X) \subseteq V(X)$ the closed subspace of G-subflows

$\mathrm{Min}_G(X) \subseteq \mathrm{Sub}_G(X)$ the minimal subflows

Ex: G ctbl. discrete $G \curvearrowright 2^G$ via $(g \cdot x)(h) = x(hg)$.

Can show $2^G \in \overline{\mathrm{Min}_G(X)}$. Similar construction for G l.c.

Ex: If G ext. am. and X a G-flow, then

$$\mathrm{Min}_G(X) = \text{set of fixed pts.}$$

What about $M(G)$ metr., $M(G)$ has comeager orb.?

Thm (Jahel - Z 2019): For G Polish grp TFAE:

1) $M(G)$ metr!_ 2) For any G-flow X, $\mathrm{Min}_G(X)$ is

$\bigvee$-closed.

Towards $(1) \Rightarrow (2)$, fix X and $(Y_i)_{i \in I}$ from $\mathrm{Min}_G(X)$.

WMA $Y_i \to Y \in \mathrm{Sub}_G(X)$. $M \cong M(G)$

Fix $\varphi_i : M \to Y_i$, view as $\varphi : \underbrace{I \times M}_{G\text{-space}} \to X$

<u>Fact (??)</u>: For Z a l.c. G-space, $\exists$ a <u>max'l G-equiv. compactification</u>

$$\begin{tikzcd} \alpha_G(Z) \arrow[dr, dotted, "\exists \tilde{\varphi}"] & \\ Z \arrow[u, hook] \arrow[r, "\psi"'] & W \text{ a } G\text{-flow} \end{tikzcd}$$

$\varphi : I \times M \to X$

On $\alpha_G(I \times M)$, we have maps:

$\tilde{\varphi} : \alpha_G(I \times M) \to X$ $\quad \pi_I : \alpha_G(I \times M) \to \beta I$ $\quad \pi_M : \alpha_G(I \times M) \to M$

If $\mathcal{U} \in \beta I$, write $M_{\mathcal{U}} = \pi_I^{-1}(\{\mathcal{U}\})$. If $\mathcal{U}$ contains tails of I, $\tilde{\varphi}[M_{\mathcal{U}}] = Y$

Idea: We should regard $M_{\mathcal{U}}$ as an ultrapower of M

Thm (Basgo-Z.) TFAE:

1) $M(G)$ has com. orb.

2) For any G-flow X, $(Y_i)_{i\in I}$ from $\mathrm{Min}_G(X)$, then if

$Y_i \longrightarrow Y \in \mathrm{Sub}_G(X)$, then Y contains a unique min. subflow.